Automatic Abstraction

Current automatic abstraction tools typically proceed as follows:

- Given a concrete program C, they generate an abstract program A such that "A simulates C".
- For any $\forall$-properties ϕ, $A \models \phi$ implies $C \models \phi$.

Limitations:

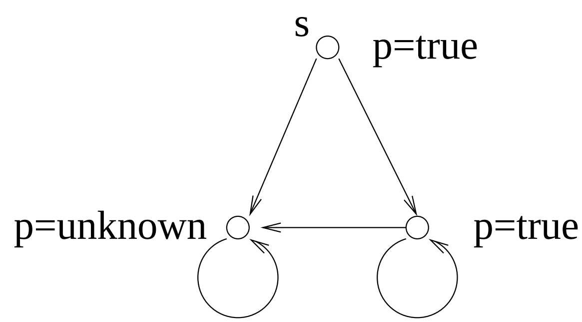
- Restricted to $\forall$-properties (no existential properties).
- $A \not\models \phi$ does not imply anything about C!
- Could the analysis be more precise for a comparable cost?

A Solution: use 3-Valued Models [Bruns-G99]

Use richer models A that distinguish what is *true*, *false* and unknown ($\perp$) of C.

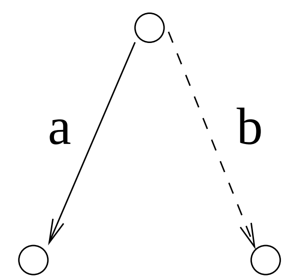
Example: partial Kripke structure (PKS) [Fitting92,Bruns-G99]

- A Kripke structure where propositions can be *true*, *false* or $\perp$.



Example: Modal Transition System [Larsen-Thomsen88]

- A LTS with $\xrightarrow{may}$ and $\xrightarrow{must}$ transitions such that $\xrightarrow{must} \subseteq \xrightarrow{may}$.



Example: Kripke Modal Transition System [Huth-Jagadeesan-Schmidt01]

- A PKS with $\xrightarrow{may}$ and $\xrightarrow{must}$ transitions such that $\xrightarrow{must} \subseteq \xrightarrow{may}$.

These models are all equally expressive [G-Jagadeesan03].

Other examples: extended transition systems [Milner81],...

3-Valued Temporal Logics

Reasoning about 3-valued models requires 3-valued TL.

Example: 3-valued Propositional Modal Logic $\phi ::= p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid AX\phi$

Semantics: (extension of Kleene's strong 3-valued PL)

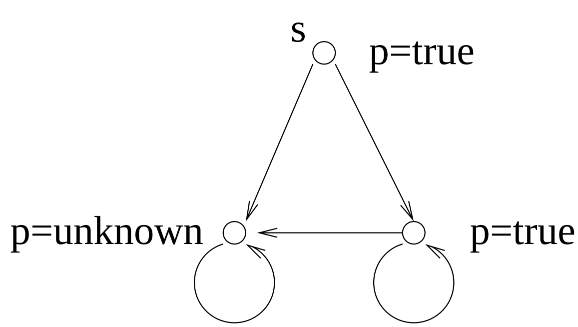
$[(M, s) \models p] = L(s, p)$

$[(M, s) \models \neg\phi] = \text{comp}([(M, s) \models \phi])$
where comp maps *true* $\mapsto$ *false*, *false* $\mapsto$ *true*, and $\bot \mapsto \bot$

$[(M, s) \models \phi_1 \wedge \phi_2] = min([(M, s) \models \phi_1], [(M, s) \models \phi_2])$
with min defined with *false* $< \bot <$ *true* ("truth" ordering)

$$[(M, s) \models AX\phi] = \begin{cases} true & \text{if } \forall s' : s \xrightarrow{may} s' \Rightarrow [(M, s') \models \phi] = true \\ false & \text{if } \exists s' : s \xrightarrow{must} s' \wedge [(M, s') \models \phi] = false \\ \bot & \text{otherwise} \end{cases}$$

- Ex: $[(M, s) \models p] = true$
- Ex: $[(M, s) \models AXp] = \bot$



Completeness Preorder

To measure the *completeness* of models (aka, *refinement* preorder, or *abstraction*$^{-1}$.)

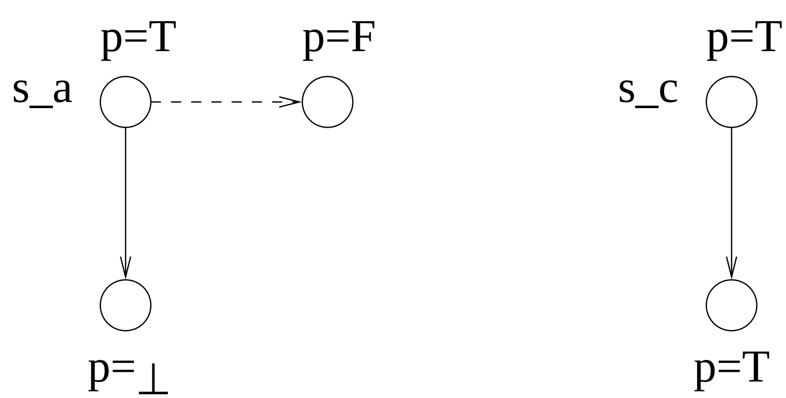
Let $\leq$ be the "information" ordering on truth values in which $\bot \leq true$ and $\bot \leq false$.

Definition: The *completeness preorder* $\preceq$ is the greatest relation $\preceq \subseteq S \times S$ such that $s_a \preceq s_c$ implies the following:

- $\forall p \in P : L_A(s_a, p) \leq L_C(s_c, p)$,
- if $s_a \xrightarrow{must}_A s'_a$, there is some $s'_c \in S_C$ such that $s_c \xrightarrow{must}_C s'_c$ and $s'_a \preceq s'_c$,
- if $s_c \xrightarrow{may}_C s'_c$, there is some $s'_a \in S_A$ such that $s_a \xrightarrow{may}_A s'_a$ and $s'_a \preceq s'_c$.

(Note: if no $\bot$ and only $\xrightarrow{may}$, $\preceq$ is simulation.)

Example:



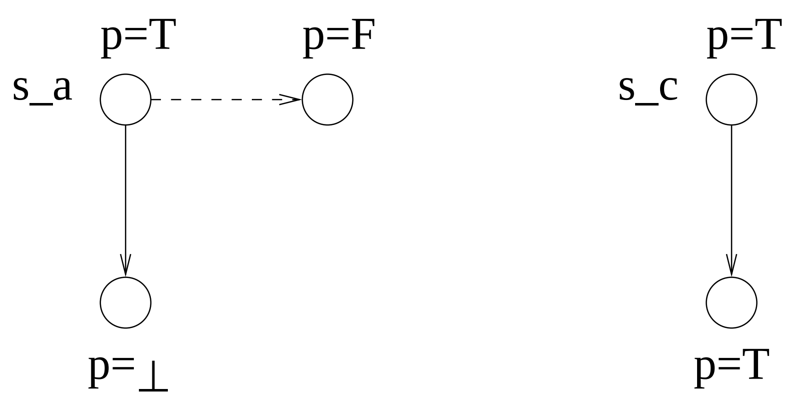
Logical Characterization of Completeness Preorder

Theorem: Let Φ denote the set of all formulas of 3-valued propositional modal logic. Then

$$s_a \preceq s_c \text{ iff } (\forall \phi \in \Phi : [s_a \models \phi] \leq [s_c \models \phi]).$$

Thus, models that are "more complete" with respect to $\preceq$ have more definite properties with respect to $\leq$.

Example:



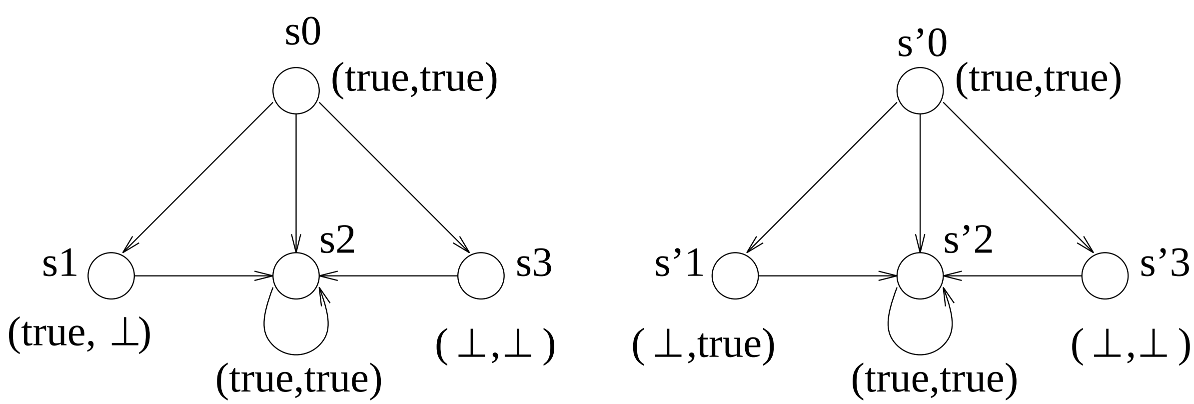
Completeness Preorder (Continued)

Corollary:
Let Φ denote the set of all formulas of 3-valued propositional modal logic. Then

$$(\forall \phi \in \Phi : [(M_1, s_1) \models \phi] = [(M_2, s_2) \models \phi]) \text{ iff}$$
$$(s_1 \preceq s_2 \text{ and } s_2 \preceq s_1).$$

Note: If s_1 and s_2 are bisimilar, this implies $s_1 \preceq s_2$ and $s_2 \preceq s_1$, but $s_1 \preceq s_2$ and $s_2 \preceq s_1$ does not imply s_1 and $s2$ are bisimilar! [Bruns-G99]

Example: s_0 and s_0' are not bisimilar, but cannot be distinguished by any formula of 3-valued propositional modal logic.



3-Valued Model Checking

Problem: Given a state s of a 3-valued model M and a formula ϕ, how to compute the value $[(M,s) \models \phi]$?

Theorem: [Bruns-G00] The model-checking problem for a 3-valued temporal logic can be reduced to two model-checking problems for the corresponding 2-valued logic.

STEP 1: complete M into two "extreme" complete Kripke structures, called the **optimistic** and **pessimistic** completions:

- Extend P to P' such that, for every $p \in P$ there exists a $\bar{p} \in P'$ such that $L(s,p) = \text{comp}(L(s,\bar{p}))$ for all s in S.
- $M_o = (S, L_o, \xrightarrow{must})$ with

$$L_o(s,p) \stackrel{\text{def}}{=} \begin{cases} true & \text{if } L(s,p) = \perp \\ L(s,p) & \text{otherwise} \end{cases}$$

- $M_p = (S, L_p, \xrightarrow{may})$ with

$$L_p(s,p) \stackrel{\text{def}}{=} \begin{cases} false & \text{if } L(s,p) = \perp \\ L(s,p) & \text{otherwise} \end{cases}$$

3-Valued Model Checking (Continued)

STEP 2: transform ϕ to its positive form $T(\phi)$ with $T(\neg p) = \bar{p}$.

STEP 3: evaluate $T(\phi)$ on M_o and M_p using traditional 2-valued model checking, and combine the results:

$$[(M, s) \models \phi] = \begin{cases} true & \text{if } (M_p, s) \models T(\phi) \\ false & \text{if } (M_o, s) \not\models T(\phi) \\ \perp & \text{otherwise} \end{cases}$$

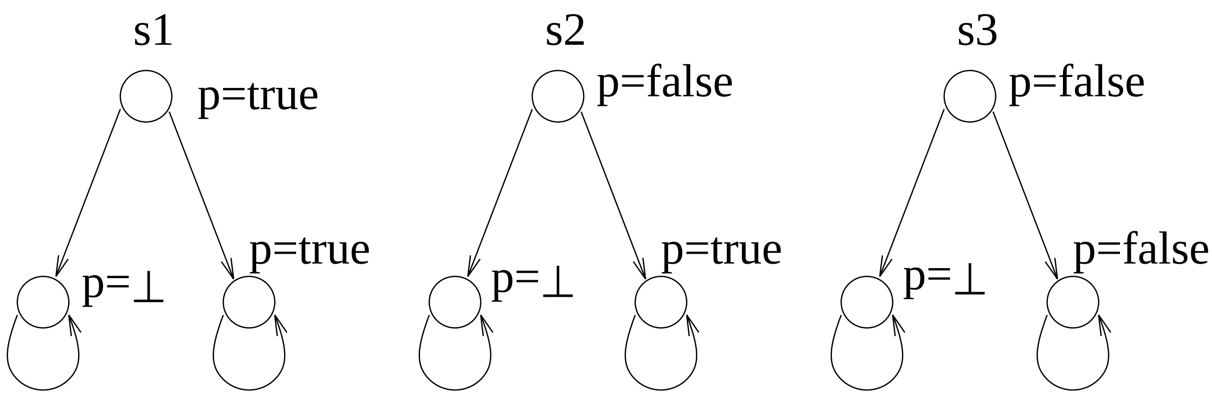
This can be done using existing model-checking tools!

Corollary: 3-valued model checking has the same complexity as traditional 2-valued model checking.

Examples

Application:
Generation of a partial Kripke structure from a partial state-space exploration such that, by construction, $s'_0 \preceq s_0$ [Bruns-G99].

Examples:



- $[s_1 \models A(true\,\mathcal{U}\,p)] = true$
- $[s_2 \models A(true\,\mathcal{U}\,p)] = \perp$
- $[s_3 \models A(true\,\mathcal{U}\,p)] = false$

New 3-Valued Semantics

Observation: One can argue that the previous semantics returns $\perp$ more often than it should...

Example: In a state s_a where $p = \perp$ and $q = true$,

$$[s_a \models \ q \wedge (p \vee \neg p) \ ] = \perp$$

while the same formula is *true* in every complete state s_c such that $s_a \preceq s_c$!

New 3-valued "thorough" semantics: [Bruns-G00]

$$[(M,s) \models \phi]_t = \begin{cases} true & \text{if } (M',s') \models \phi \text{ for all } (M',s') : s \preceq s' \\ false & \text{if } (M',s') \not\models \phi \text{ for all } (M',s') : s \preceq s' \\ \perp & \text{otherwise} \end{cases}$$

Is model checking more expensive with this semantics?

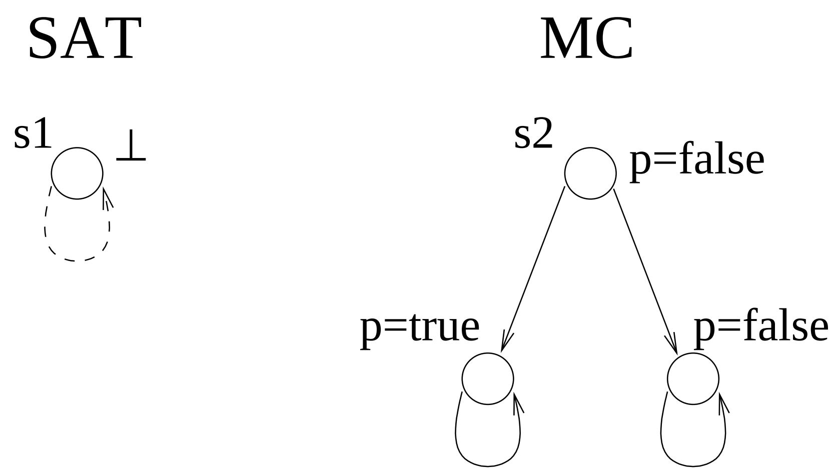
YES! Indeed, in general, one needs to solve two

Generalized Model-Checking Problems

Generalized Model Checking [Bruns-G00]

Definition: Given a state s of a model M and a formula ϕ of a temporal logic L, is there a state s' of a complete system M' such that $s \preceq s'$ and $(M', s') \models \phi$?

This **generalized model-checking problem** is thus a generalization of both **satisfiability** (all Kripke structures are potential solutions) and **model checking** (a single Kripke structure needs to be checked).



Theorem: The satisfiability problem for a temporal logic L is reducible (in linear-time and logarithmic space) to the generalized model-checking problem for L.

Thus, GMC is as hard as satisfiability. Is it harder?

Branching-Time Temporal Logics

Theorem: (CTL) Given a state s_0 of partial Kripke structure $M = (S, L, \mathcal{R})$ and a CTL formula ϕ, one can construct an alternating Büchi word automaton $A_{(M,s_0),\phi}$ over a 1-letter alphabet with at most $O(|S| \cdot 2^{O(|\phi|)})$ states such that

$$(\exists (M', s_0') : s_0 \preceq s_0' \text{ and } (M', s_0') \models \phi) \text{ iff } \mathcal{L}(A_{(M,s_0),\phi}) \neq \emptyset.$$

Corollary: if such a M' exists, there exists one with at most $|S| \cdot 2^{O(|\phi|)}$ states.

Theorem: The generalized model-checking problem for a state s_0 of a partial Kripke structure $M = (S, L, \mathcal{R})$ and a CTL formula ϕ can be decided in time $O(|S|^2 \cdot 2^{O(|\phi|)})$.

Theorem: The generalized model-checking problem for CTL is EXPTIME-complete.

Theorem: (Summary) Let L denote propositional logic, propositional modal logic, CTL, or any branching-time logic including CTL (such as CTL* or the modal μ-calculus). The generalized model-checking problem for the logic L has the same complexity as the satisfiability problem for L.

Linear-Time Temporal Logics

Theorem: (LTL) Given a state s_0 of partial Kripke structure $M = (S, L, \mathcal{R})$ and an LTL formula ϕ, one can construct an alternating Büchi word automaton $A_{(M,s_0),\phi}$ over a 1-letter alphabet with at most $O(|S| \cdot 2^{|\phi|})$ states such that

$$(\exists (M', s'_0) : s_0 \preceq s'_0 \text{ and } (M', s'_0) \models \phi) \text{ iff } \mathcal{L}(A_{(M,s_0),\phi}) \neq \emptyset.$$

Theorem: The generalized model-checking problem for a state s_0 of a partial Kripke structure $M = (S, L, R)$ and an LTL formula ϕ can be decided in time $O(|S|^2 \cdot 2^{2|\phi|})$.

Theorem: The generalized model-checking problem for linear-time temporal logic is EXPTIME-complete.

For LTL, generalized model checking is thus **harder** than satisfiability and model checking! [Bruns-G00]

(both of these problems are PSPACE-complete for LTL)

Note: similar phenomenon for "realizability" and "synthesis" for LTL specifications [Abadi-Lamport-Wolper89, Pnueli-Rosner89].

Summary on Complexity in $|\phi|$

Model Checking: (3-valued semantics)

- MC can be reduced to two 2-valued MC problems.
- MC has the same complexity as 2-valued MC.

Generalized Model Checking: (thorough 3-val. sem.)

- For BTL, GMC has the same complexity as satisfiablity.
- For LTL, GMC is harder than satisfiablity and MC.

Logic	**MC**	**SAT**	**GMC**
PL	Linear	NP-Complete	NP-Complete
PML	Linear	PSPACE-Complete	PSPACE-Complete
CTL	Linear	EXPTIME-Complete	EXPTIME-Complete
μ-calculus	NP$\cap$co-NP	EXPTIME-Complete	EXPTIME-Complete
LTL	PSPACE-Complete	PSPACE-Complete	EXPTIME-Complete

Complexity of GMC in $|M|$

Upper bound: can be done in quadratic time in $|M|$ [Bruns-G00].

Theorem: [G-Jagadeesan02] Checking emptiness of nondeterministic Büchi tree automata is reducible (in linear time and logarithmic space) to GMC for LTL (or CTL) properties represented by nondeterministic Büchi word (resp. tree) automata.

Bad News: (Lower bound) The best algorithm known for checking emptiness of nondeterministic Büchi tree automata A requires quadratic time in $|A|$ in the worst case [Vardi-Wolper86].

Good News: better complexity for GMC and properties recognizable by nondeterministic *co-Büchi* word/tree automata, i.e., *persistence properties* (e.g., LTL formulas of the form $\Diamond \Box p$).

Theorem: [G-Jagadeesan02] GMC for persistence properties can be solved in time linear in $|M|$.

Note: persistence properties include all safety ($\Box p$) and guarantee ($\Diamond p$) properties. (Do not include $\Box \Diamond p$.)

Application: Automatic Abstraction

Idea: Given a concrete system C, if $C \models \phi$ cannot be decided, generate a (smaller) abstraction A and check $A \models \phi$ instead.

Example: predicate abstraction

- Let $\psi_1, \ldots, \psi_n$ be n predicates on variables of C.
- Abstract states are vectors of n bits b_i.
- A concrete state c is abstracted by an abstract state

$$[c] = (b_1, \ldots, b_n) \text{ iff } \forall 1 \leq i \leq n : b_i = \psi_i(c).$$

State of the art: A is a traditional 2-valued model with

$$(c_1 \to c_2) \Rightarrow ([c_1] \to [c_2]).$$

In other words, A **simulates** C. Remember, this implies:

- If ϕ is a $\forall$-property, $A \models \phi$ implies $C \models \phi$,
- but $A \not\models \phi$ does not imply anything about C!

Automatic Abstraction Revisited

Observation: A should really be a 3-valued model!

For instance, A can be represented by a modal transition system.

Abstraction relation:

1. $(c_1 \rightarrow c_2) \Rightarrow ([c_1] \rightarrow_{\text{may}} [c_2])$
2. $(\forall c_i \in [a] : \exists c_i \rightarrow c_j \wedge c_j \in [a']) \Rightarrow ([a] \rightarrow_{\text{must}} [a'])$

By construction, $A \preceq C$.

Computing an MTS A using (1)+(2) can be done at the same computational cost (same complexity) as computing a "conservative" abstraction (simulation) using (1) alone: (2) can be built by dualizing all the steps necessary to build (1).

This is shown for predicate and cartesian abstraction in [G-Huth-Jagadeesan01].

Automatic Abstraction Process

Traditional iterative abstraction procedure:

1. Abstract: compute M_A that simulates M_C.
2. Check: given a universal property ϕ, check $M_A \models \phi$.
   - if $M_A \models \phi$: stop (the property is proved: $M_C \models \phi$).
   - if $M_A \not\models \phi$: go to Step 3.
3. Refine: refine M_A. Then go to Step 1.

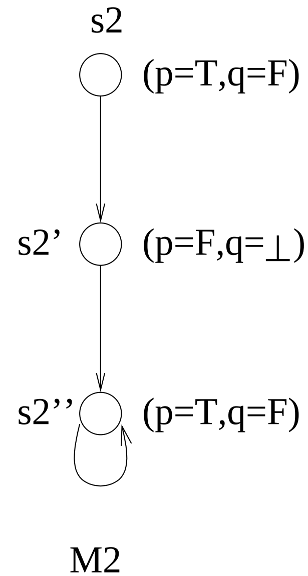
New procedure for automatic abstraction: (3 improvements)

1. Abstract: compute M_A such that $M_A \preceq M_C$ (same cost as above [GHJ01])
2. Check: given *any* property ϕ,
   1. (3-valued model checking) compute $[M_A \models \phi]$.
      - if $[M_A \models \phi] = \mathit{true}$ or false: stop .
      - if $[M_A \models \phi] = \perp$, continue.
   2. (generalized model checking) compute $[M_A \models \phi]_t$.
      - if $[M_A \models \phi]_t = \mathit{true}$ or false: stop .
      - if $[M_A \models \phi]_t = \perp$, go to Step 3.
3. Refine: refine M_A. Then go to Step 1.

Example

Predicate abstraction with p : "is x odd?" and q : "is y odd?" such that $M_2 \preceq C_2$:

```
program C2() {
  x,y = 1,0;
  x,y = 2*f(x),f(y);
  x,y = 1,0;
}
```



For $\phi_2 = \Diamond q \wedge \Box(p \vee \neg q)$, $[(M_2, s_2) \models \phi_2] = \perp$, but $[(M_2, s_2) \models \phi_2]_t = \mathit{false}$ (i.e., there does not exist a concretization of (M_2, s_2) that satisfies ϕ_2).

Thus, GMC is more precise than MC in this case.

(Same for the safety property $\phi_2' = \bigcirc q \wedge \Box(p \vee \neg q)$.)

Precision of GMC Vs. MC

How often is GMC more precise than MC? See [G-Huth05]:

- Studies when it is possible to reduce GMC(M, ϕ) to MC(M, ϕ').
- ϕ' is called a *semantic minimization* of ϕ.
- Shows that PL (already known), PML, and μ-calculus are closed under semantic minimization, but not LTL, CTL or CTL*.
- Identifies *self-minimizing* formulas, i.e., ϕ's for which GMC$(M, \phi) =$ MC(M, ϕ)
  - semantically (using automata-theoretic techniques, EXPTIME-hard in $|\phi|$ for μ-calculus) and
  - syntactically (sufficient criterion only, linear in $|\phi|$).
- Ex (syntactic): Any formula that does not contain any atomic proposition in mixed polarity (in its negation normal form) is self-minimizing.
- Note: the converse is not true (e.g., $(\neg q_1 \vee q_2) \wedge (\neg q_2 \vee q_1)$ is self-minimizing).
- For any self-minimizing formula, GMC and MC have the same precision.
- Good news: many frequent formulas of practical interest are self-minimizing, and MC is as precise as GMC for those.

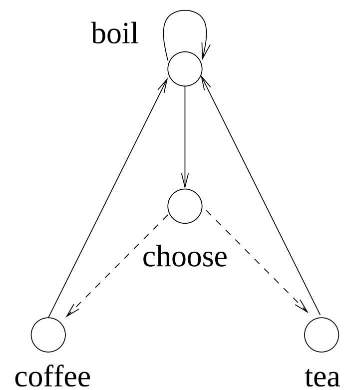
3-Valued Abstractions for Open Systems

Open system: system interacting with its environment.

Module Checking (ModC) [Kupferman-Vardi96]: given an open system M and a formula ϕ, does M satisfy ϕ in *all possible environments*?

Example: (vending machine)
is it always possible for M to eventually serve tea?

- MC(M, AGEF tea) $= true$
- ModC(M, AGEF tea) $= false$!



Generalized Module Checking (GModC) [G03]: given A and ϕ, does there exist a concretization C of A such that C satisfies ϕ in all possible environments?

Two simulataneous games here: one with the environment, one with $\perp$ values...

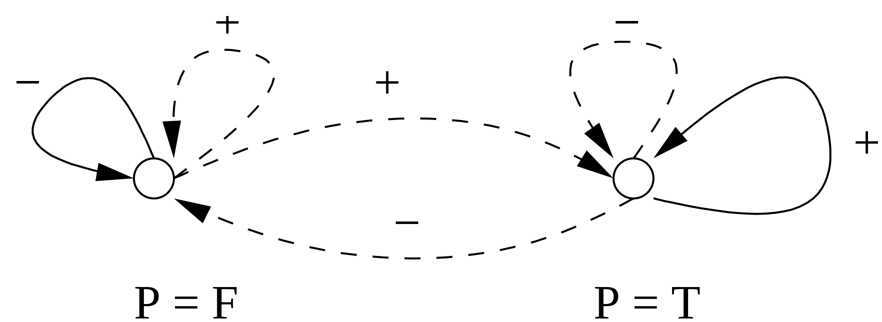
Yet, GModC can be solved at the same cost as GMC (for LTL and BTL) [G03].

3-Valued Abstractions for Games

Study abstractions of games where moves of each player can now be abstracted, while preserving winning strategies of *both* players [de Alfaro-G-Jagadeesan04]:

- An abstraction of a game is now a game where each player has both may and must moves (yielding may/must strategies).
- Completeness preorder is now an *alternating refinement* relation, logically characterized by *3-valued alternating μ-calculus* [Alur-Henzinger-Kupferman02].
- If must transitions are allowed to be *nondeterministic* [Larsen-Xinxin90], then the abstraction is as precise as can be, i.e., the framework is *complete* (see also [Namjoshi03, Dams-Namjoshi04]):
  "Given any infinite-state system C and property $\phi \in \mu$-calculus, if C satisfies ϕ, then there exists a finite-state abstraction A such that A satisfies ϕ."

  Example: [Namjoshi03]
  - var x;
    actions (-) x:=x-1; (+) x:=x+1;
  - property: EF(P) with P= (x$\geq$ 0)



- The construction of abstraction is now *compositional* (cf. [G-Huth-Jagadeesan01], [Shoham-Grumberg04], [de Alfaro-G-Jagadeesan04]).

Conclusions

3-Valued models and logics can be used to check *any property*, while *guaranteeing soundness of counter-examples*.

Generalized Model Checking means checking whether there exists a concretization of an abstraction that satisfies a temporal logic formula.

It can be used to improve precision of automatic abstraction, for a reasonable cost:

- Cost can be higher in the size of the formula...
  but only worst-case and formulas are short.
- Cost can be higher (quadratic) in the size of the model...
  but is the same (linear) for persistence properties (includes safety).

In an "abstract-check-refine" procedure, GMC is only polynomial in the size of the abstraction, and may prevent the unnecessary generation and analysis of possibly exponentially larger refinements of that abstraction.

In practice, use first a syntactic formula check for self-minimization:
MC has then the same precision as GMC (often the case).

Other Related Work

"Mixed transition systems" [Dams-Gerth-Grumberg94]

- Intuitively, a mixed transition system is an MTS without the constraint $\xrightarrow{must} \subseteq \xrightarrow{may}$.
- Hence, more expressive than 3-valued models: some mixed TS cannot be refined into any complete system.
- Still, their goal is very similar (i.e., design may/must abstractions for MC).

"Extended transition systems" [Milner81]

- XTS = LTS + "divergence predicate"
- In [Bruns-G99], it is shown that 3-valued Hennessy-Milner Logic logically characterizes the "divergence preorder" [Milner81,Walker90].
- Close correspondence with Plotkin's intuitionistic modal logic (inspired Bruns-G00 reduction from 3-val to 2-val MC).

3-Valued logic for program analysis: [Sagiv-Reps-Wilhelm99] shape graphs, first-order 3-valued logic, "focussing",... (roughly inspired the beginning of this work but technical details are fairly different – e.g., no 3-valued abstraction on control)

Conservative abstraction for the full mu-calculus: [Saidi-Shankar99]